Oblivious Points on Translation Surfaces

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Abstract

An **oblivious point** is a point on a **translation surface** contained in **no closed geodesic**. Nguyen, Pan, and Su (2018) proved that there are only **finitely many** oblivious points on any translation surface. We:

- **restrict** which points can be oblivious on surfaces tiled by a regular polygon
- devise an algorithm to **determine obliviousness**
- construct surfaces with **arbitrarily many** oblivious points
- prove **every genus** ≥ 3 admits an oblivious point

Background

- **Translation surface:** polygons in the plane with parallel sides identified by translation
- **Closed geodesic:** locally distance minimizing closed curve
- **Cone angles** of a translation surface are integer multiples of 2π.
- **Gauss-Bonnet**: $\Sigma_i(\theta_i - 2\pi) = 2\pi(2g-2)$

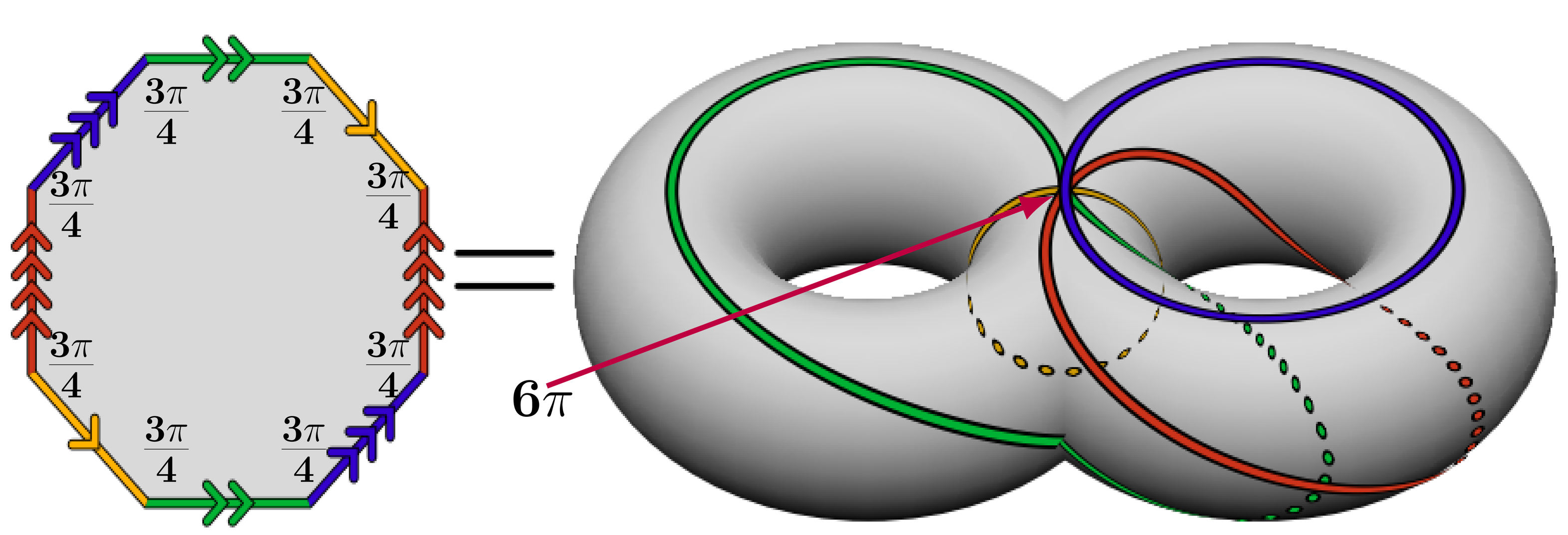

Fig 1: Translation surface with cone point

Which Points Can Be Oblivious

Lemma 1 (DJZ '19)

On a surface tiled by regular polygons, all oblivious points are vertices. Thus, the only regular-polygon-tiled surfaces that admit oblivious points are tiled by triangles, squares, and hexagons.

Proposition 1 (DJZ '19)

On a square-tiled surface, there exists a finite time check to determine if a point is oblivious.

Previous Results

Semi-translation surface: polygons in the place with sides identified by translation or reflection $z \mapsto \pm z + c \in \mathbb{C}$. Cone angles are integer multiples of π

Proposition 2 (Nguyen, Pan, Su '18)

Let Q be a semi-translation surface with exactly one cone point p of cone angle π. The canonical double cover of Q has an oblivious point at the pre-image of p.

Corollary 1 (DJZ '19)

There exists a square-tiled surface with an even number of squares containing an oblivious point for all even integers ≥ 8

Slit Construction for Oblivious Points

Lemma 2 (DJZ '19)

The pre-images of an oblivious point under a translation cover are oblivious.

- **Blocking Set:** A finite set of points so that any geodesic originating from a given point will either hit the blocking set or never close

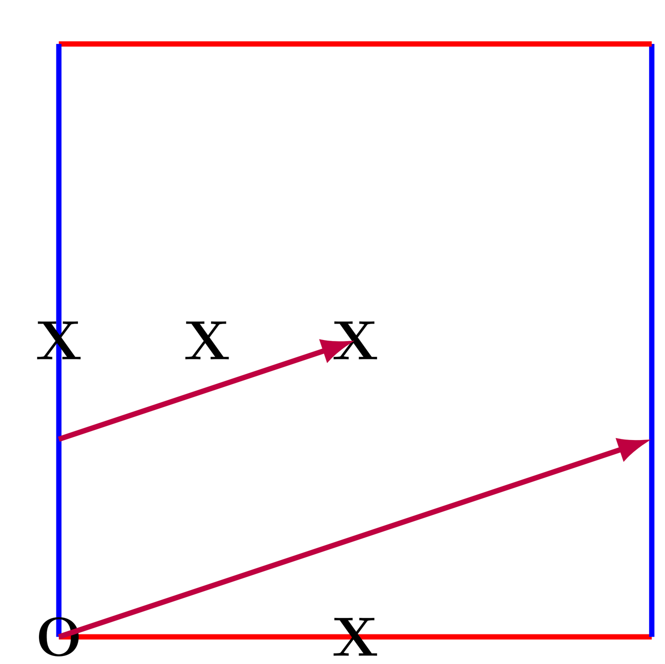

Fig 2: Blocking set on a square torus

Our Construction

- Take n square tori with marked points at a blocking set
- Slit between the blocking points to create a connected surface
- Each blocking point becomes a cone point of angle $2\pi n$
- By Lemma 2, every blocked point is oblivious

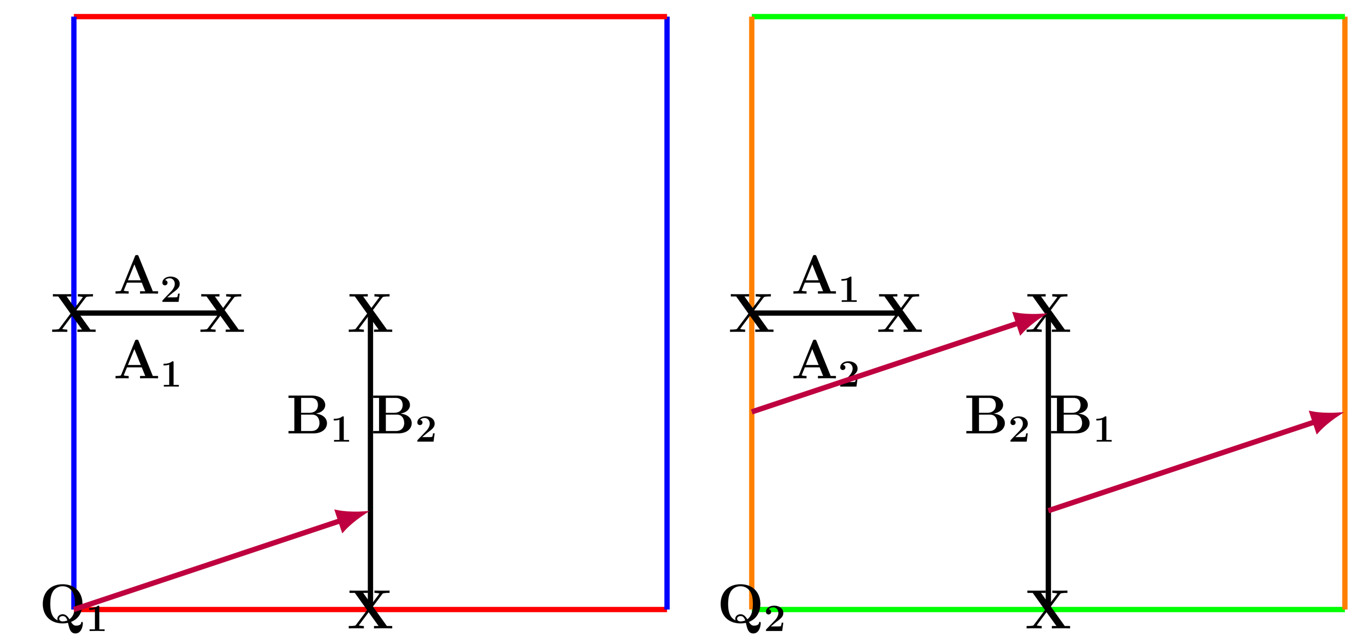

Fig. 3: The construction for n=2, Q_1 and Q_2 are oblivious

Results

Theorem 1 (DJZ '19)

For every $n \in \mathbb{N}$, there is a connected surface with n oblivious points.

Proposition 3 (DJZ '19)

There exists a square-tiled surface with an oblivious point for surfaces tiled by kn^2 squares, $k \geq 2, n \geq 2$

- Our construction implies there is a surface in every genus $2n-1, n \geq 2$ with oblivious points
- Masur and Smillie (1986) proved that $\mathcal{Q}(-1, 4n+1), n \geq 2$ is nonempty
- Applying Proposition 2, this implies there is a surface with oblivious points in every genus $2n-2, n \geq 2$

Theorem 2 (DJZ '19)

There is a square-tiled surface in every genus $g \geq 3$ which admits an oblivious point.

References and Acknowledgements

- [1] Nguyen, Pan, Su, *Existence of closed geodesics through a regular point on translation surfaces*, arXiv:1712.09066v2, 2018.
- [2] Lelievre, Monteil, Weiss, *Everything is Illuminated*, arXiv:1407.2975, 2015
- [3] Masur and Smillie, *Ergodicity of Billiard Flows and Quadratic Differentials*, Ann. Math.(2)123-456, 1986
- [4] math.stackexchange.com/questions/2474295/

The authors would like to thank Aaron Calderon and Ian Adelstein, whose dedicated mentorship made this project possible